

Exact joint laws associated with spectrally negative Lévy processes and applications to insurance risk theory

Chuancun YIN

School of Mathematical Sciences, Qufu Normal University
Shandong 273165, China

Kam C. YUEN

Department of Statistics and Actuarial Science
The University of Hong Kong, Pokfulam, Hong Kong, China

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Abstract We consider the spectrally negative Lévy processes and determine the joint laws for the quantities such as the first and last passage times over a fixed level, the overshoots and undershoots at first passage, the minimum, the maximum and the duration of negative values. We apply our results to insurance risk theory to find an explicit expression for the generalized expected discounted penalty function in terms of scale functions. Further, a new expression for the generalized Dickson's formula is provided.

Keywords Fluctuation identity, spectrally negative Lévy processes, generalized Dickson's formula, scale functions, occupation times, Suprema and infima
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1 Introduction

Let $X = \{X(t), t \geq 0\}$ be a real valued spectrally negative Lévy process, i.e. a stochastic process with càdlàg paths without positive jumps that has stationary independent increments defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ where the filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfies the usual conditions of right continuity and completion. The reader is referred to Bertoin [5], Kyprianou [25] and Doney [12] for a general discussion to Lévy processes. Write P_x for the probability law of X when $X(0) = x$ and E_x for the expectation with respect to P_x . We simply

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Corresponding author: Chuancun YIN, E-mail: ccyin@mail.qfnu.edu.cn

write $P_0 = P$ and $E_0 = E$. As usual, we will exclude the case that X has monotone paths. By the absence of positive jumps, the moment generating function of $X(t)$ exists for all $\alpha \geq 0$ and is given by $Ee^{\alpha X(t)} = e^{t\varphi(\alpha)}$, $t \geq 0$, where $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is known as the Laplace exponent. It is given by the Lévy-Khinchin formula

$$\varphi(\alpha) = \mu\alpha + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^0 (e^{\alpha x} - 1 - \alpha x 1_{(x>-1)})\Pi(dx), \quad (1.1)$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(-\infty, 0)$ known as the Lévy measure satisfying $\int_{-\infty}^0 (x^2 \wedge 1)\Pi(dx) < \infty$. It is known that $\varphi(x)$ is strictly convex for $x \geq 0$ and that $\varphi(0) = 0$ and $\lim_{x \rightarrow \infty} \varphi(x) = \infty$. Moreover, $\varphi'(0+) = E(X(1)) = \mu + \int_{-\infty}^{-1} x\Pi(dx) \in [-\infty, \infty)$. Recall that X drifts to $-\infty$ if and only if $\varphi'(0+) < 0$. Such a spectrally negative Lévy process has sample paths of bounded variation if and only if $\sigma = 0$ and $\int_{-1}^0 |x|\Pi(dx) < \infty$.

Define the right inverse $\Phi(q) = \sup\{\alpha \geq 0 : \varphi(\alpha) = q\}$ for each $q \geq 0$. If $\varphi'(0+) \geq 0$ then $\alpha = 0$ is the unique solution to $\varphi(\alpha) = 0$ and otherwise there are two solutions with $\alpha = \Phi(0) > 0$ being the larger of the two, the other is $\alpha = 0$. The Laplace exponent φ is continuous and increasing over $[\Phi(0), \infty)$, so that $\lim_{\alpha \rightarrow 0+} \Phi(\alpha) = \Phi(0)$.

It is known that (see Ref. [3]) for any c such that $\varphi(c) = \log E[\exp cX(1)]$ is finite, the family $\Lambda_t(c) = \exp\{c(X(t) - x) - \varphi(c)t\}$ is a martingale under P_x . Let $P_x^{(c)}$ denote the probability measure on $\mathcal{F} = \sigma\{X(s) : 0 \leq s < \infty\}$ defined by

$$\left. \frac{dP_x^{(c)}}{dP_x} \right|_{\mathcal{F}_t} = \frac{\Lambda_t(c)}{\Lambda_0(c)} \quad (1.2)$$

for all $0 \leq t < \infty$. Under the measure $P_x^{(c)}$, X remains within the class of spectrally negative process and the Laplace exponent of X is given by $\varphi_c(\alpha) = \varphi(\alpha + c) - \varphi(c)$, $\alpha \geq -c$.

When we set $c = \Phi(q)$ for $q \geq 0$ we find that $\varphi'_{\Phi(q)}(0) = \varphi'(\Phi(q)) \geq 0$. Thus, $E^{(\Phi(q))}(X(1)) = \varphi'(\Phi(q)) > 0$ for $q > 0$. In particular, under the measure $P^{(\Phi(q))}$, X always drifts to ∞ for $q > 0$ (see Kyprianou [25, pp 213-214] for more details).

Denote by I and S the past infimum and supremum of X respectively, that is, $I_t = \inf_{0 \leq s \leq t} X(s)$ and $S_t = \sup_{0 \leq s \leq t} X(s)$. Define the first passage times above and below a for X by $T_a^- = \inf\{t \geq 0 : X(t) < a\}$ and $T_a^+ = \inf\{t \geq 0 : X(t) > a\}$. We simply write T for T_0^- . Finally, let T_0 denote the time of recovery: $T_0 = \inf\{t : t > T, X(t) = 0\}$.

The fluctuation theory for spectrally negative Lévy processes has been the object of several studies over the last 40 years, among many others, see Emery [15], Bingham [7] and Bertoin [6]; We refer to Kyprianou and Palmowaki [26] and Pistorius [32] for exhaustive reviews. In a great variety of problems of applied fields such as actuarial mathematics, mathematical finance and queueing

theory and so on one faces the consideration of the fluctuation theory for a class of spectrally negative Lévy processes; For example, The exit problems have recently been used by Avram et al. [3] in the context of finance in connection with American and Canadian options. Alili and Kyprianou [1] provided, with the help of a fluctuation identity, a generic link between a number of known identities for the first passage time and overshoot above/below a fixed level of a Lévy process to the American perpetual put optimal stopping problem. In the theory of actuarial mathematics, the first passage problem is very important to the ruin problems; see, for example, among others, Yang and Zhang [35], Klüppelberg et al. [23], Huzak et al. [20], Zhou [36], Garrido and Morales [17], Biffis and Kyprianou [8] and Billis and Morales [9].

Several recent papers have concerned the joint laws of overshoots and undershoots of Lévy processes at the first and the last passage times of a constant barrier. For example, Doney and Kyprianou [13] studied the problem for general Lévy process X . They computed the quintuple law of the time of the first passage relative to the time of the last maximum at the first passage, the time of the last maximum at the first passage, the overshoot at the first passage, the undershoot at the first passage and the undershoot of the last maximum at the first passage, i.e. the law of $(T_x^+ - \overline{G}_{T_x^+}, \overline{G}_{T_x^+}, X_{T_x^+} - x, x - X_{T_x^+}, x - S_{T_x^+})$, where $\overline{G}_t = \sup\{s < t, X_s = S_s\}$. Kyprianou, Pardo and Rivero (2010) extended above quintuple law to a family of related joint laws for Lévy processes, Lévy processes conditioned to stay positive and positive self-similar Markov processes at both first and last passage over a fixed level. Eder and Klüppelberg [16] derived new results in fluctuation theory for sums of possibly dependent Lévy processes. Chaumont, Kyprianou and Pardo [10] consider some special classes of Lévy processes with no Gaussian component whose Lévy measure is of the type $\Pi(dx) = \pi(x)dx$, where $\pi(x) = e^{\gamma x}\nu(e^x - 1)$.

Motivated by these interesting papers, we continue to study the fluctuation theory of Lévy processes. In this paper, we restrict ourselves to the spectrally negative Lévy processes. The advantage is that all results can be explicitly expressed in terms of scale functions. The rest of the paper is organized as follows. The next section reviews some preliminary results of spectrally negative Lévy processes that will be needed later on. In Section 3 we determine the joint laws of all or some of the quantities such as the first and last passage time over a fixed level, the overshoots and undershoots at the first passage, the minimum, the maximum and the duration of negative values. Applications in insurance risk theory are discussed in Section 4.

2 Preliminaries

Let us now reviewing some preliminary results of spectrally negative Lévy processes. We will assume that the measure Π has a density π , with respect to the Lebesgue measure. So that the scale functions are differentiable. For details, see Ref. [29].

2.1 The scale functions and the survival probabilities

Scale functions are key object in the fluctuation theory of spectrally negative Lévy processes and its applications, the survival probabilities are key object in risk theory.

Definition 2.1 For $q \geq 0$, the q -scale function $W^{(q)} : (-\infty, \infty) \rightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\alpha x} W^{(q)}(x) dx = \frac{1}{\varphi(\alpha) - q}, \quad \alpha > \Phi(q)$$

and is defined to be identically zero for $x < 0$. For short we shall write $W^{(0)} = W$. Further, we shall use the notation $W_c^{(q)}(x)$ to mean the q -scale function as defined above for $(X, P^{(c)})$.

For $q \geq 0$, we define $Z^{(q)}(x) = 1$ for $x \leq 0$ and

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy \text{ for } x > 0.$$

For $x \geq 0$, define the survival probability

$$\overline{Q^{(c)}}(x) = 1 - Q^{(c)}(x) = P_x^{(c)}(I_\infty \geq 0),$$

where $\overline{Q^{(c)}}(0) = \lim_{x \downarrow 0} \overline{Q^{(c)}}(x)$. When $c = 0$, we write $\overline{Q}(x)$ instead of $\overline{Q^{(0)}}(x)$.

It is well known that (see Ref. [26]), under the condition $\varphi'(0+) > 0$, $W(x) = \frac{1}{\varphi'(0+)} P_x(I_\infty \geq 0)$. For $q > 0$ ($\varphi'(0+) > 0$ is not required), using the fact that $W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x)$, we also have

$$W^{(q)}(x) = \frac{1}{\varphi'(\Phi(q))} e^{\Phi(q)x} \overline{Q^{(\Phi(q))}}(x). \quad (2.1)$$

2.2 The triple law of T , $X(T-)$ and $X(T)$

We denote the q -potential measure of X killed on exiting $[0, \infty)$ with starting point x by $U^{(q)}(x, dy)$. That is $U^{(q)}(x, dy) = dy \int_0^\infty e^{-qt} P_t(x, y) dt$, for $q \geq 0$ with $U^{(0)} = U$, where $P_t(x, y) dy = P_x(T > t, X(t) \in dy)$. If a density of $U^{(q)}(x, dy)$ exists with respect to the Lebesgue measure for each $x \geq 0$ then we call it the potential density and denoted it by $u^{(q)}(x, y)$ (with $u^{(0)} = u$). It is well known that (see Refs. [5, 12, 25])

$$u^{(q)}(x, y) = W^{(q)}(x) e^{-\Phi(q)y} - W^{(q)}(x - y). \quad (2.2)$$

For $X(0) = x \geq 0$, let

$$f(y, z, t|x) dy dz dt = P_x(T \in dt, X(T-) \in dy, |X(T)| \in dz). \quad (2.3)$$

For $q \geq 0$, define

$$f_q(y, z|x) = \int_0^\infty e^{-qt} f(y, z, t|x) dt, \quad f_q(y|x) = \int_0^\infty f_q(y, z|x) dz.$$

It follows from Doney [12, Remark 5(i), p105] that (by letting $a \rightarrow \infty$) for $x, y, z > 0$,

$$E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz) = u^{(q)}(x, y) \pi(-z - y) dz dy, \quad (2.4)$$

and

$$\int_0^\infty e^{-qt} f(y, z, t|x) dt = \pi(-z - y) \int_0^\infty e^{-qt} P_t(x, y) dt.$$

So that

$$f_q(y, z|x) = u^{(q)}(x, y) \pi(-z - y), \quad (2.5)$$

and

$$f(y, z, t|x) = P_t(x, y) \pi(-z - y).$$

From (2.1), (2.2) and (2.5) we obtain for $y > 0, x > 0$,

$$\begin{aligned} f_q(y|x) &= (\Pi(-\infty, -y))(W^{(q)}(x)e^{-\Phi(q)y} - W^{(q)}(x - y)) \\ &= (\Pi(-\infty, -y)) \frac{e^{\Phi(q)(x-y)}}{\varphi'(\Phi(q))} \left(\overline{Q^{(\Phi(q))}}(x) - \overline{Q^{(\Phi(q))}}(x - y) \right). \end{aligned} \quad (2.6)$$

If the paths of X are of bounded variation, then by (2.5) and (2.6)

$$f_q(y, z|0) = b^{-1} e^{-\Phi(q)y} \pi(-z - y), \quad f_q(y|0) = (\Pi(-\infty, -y)) b^{-1} e^{-\Phi(q)y} \quad (2.7)$$

since $W^{(q)}(0+) = b^{-1}$, where $b = \mu - \int_{-1}^0 x \Pi(dx)$. It follows from (2.5)-(2.7) that

$$\frac{f_q(y, z|x)}{f_q(y, z|0)} = \frac{f_q(y|x)}{f_q(y|0)} = \begin{cases} bW^{(q)}(x), & 0 \leq x \leq y, \\ b(W^{(q)}(x) - e^{\Phi(q)y} W^{(q)}(x - y)), & x \geq y > 0. \end{cases} \quad (2.8)$$

Finally, for $q \geq 0$, define $\Phi_c(q)$ to be the largest real root of the equation $\varphi_c(\theta) = q$. Then we have the following important result which is due to Emery [15]; See also Kyprianou [25].

Lemma 2.1. *For any $\alpha \geq 0$ and $\beta \geq 0$, the joint Laplace transform of T_y^- and $X(T_y^-)$, with the initial condition $X(0) = x > y$, is given by*

$$E_x(e^{-\alpha T_y^- + \beta X(T_y^-)}, T < \infty) = e^{\beta x} \left(Z_\beta^{(p)}(x - y) - \frac{p}{\Phi_\beta(p)} W_\beta^{(p)}(x - y) \right), \quad (2.9)$$

where $W_\beta^{(p)}$ and $Z_\beta^{(p)}$ are scale functions with respect to the measure $P^{(\beta)}$, $p = \alpha - \varphi(\beta)$, $\Phi_\beta(p) = \Phi(\alpha) - \beta$ and $\frac{p}{\Phi_\beta(p)}$ is understood in the limiting sense if $p = 0$.

3 Joint laws for the spectrally negative Lévy processes

The main purpose of this section is to investigate some joint laws for the spectrally negative Lévy process involving some or all of the first passage time, the last passage time, the overshoots and undershoots at first passage, the minimum, the maximum and the duration of negative values.

Theorem 3.1. *For $q \geq 0$ and for positive numbers x, y, z, a and b such that $b < a \wedge x \wedge y$, $a > x, a > y$,*

$$\begin{aligned} E_x & (e^{-qT}, X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, S_{T-} \leq a, T < \infty) \\ &= \pi(-z - y) \left(\frac{W^{(q)}(x - b)}{W^{(q)}(a - b)} W^{(q)}(a - y) - W^{(q)}(x - y) \right) dydz. \end{aligned} \quad (3.1)$$

Proof Using the spatial homogeneity and the strong Markov property of $X(t)$, we obtain

$$\begin{aligned} & P_x(X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, S_{T-} > a, T < \infty) \\ &= P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, I_{T-} > 0, X(T) \leq -b, S_{T-} > a - b, T < \infty) \\ &= P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, S_{T-} > a - b, T < \infty) \\ &\quad - P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, X(T) > -b, S_{T-} > a - b, T < \infty) \\ &= P_{x-b}(S_{T-} > a - b) P_{a-b}(X(T-) \in dy - b, |X(T)| \in dz + b, T < \infty) \\ &= \frac{W(x - b)}{W(a - b)} f_0(y - b, z + b | a - b) dydz, \end{aligned}$$

$$\begin{aligned} & P_x(X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, T < \infty) \\ &= P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, I_{T-} > 0, X(T) \leq -b, T < \infty) \\ &= P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, T < \infty) \\ &\quad - P_{x-b}(X(T-) \in dy - b, |X(T)| \in dz + b, X(T) > -b, T < \infty) \\ &= f_0(y - b, z + b | x - b) dydz, \end{aligned}$$

where

$$\begin{aligned} f_0(y, z | a) dydz &= P_a(X(T-) \in dy, |X(T)| \in dz, T < \infty) \\ &= \pi(-z - y) (e^{-\Phi(0)y} W(a) - W(a - y)) dydz. \end{aligned}$$

It follows that

$$\begin{aligned} P_x & (X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, S_{T-} \leq a, T < \infty) \\ &= \pi(-z - y) \left(\frac{W(x - b)}{W(a - b)} W(a - y) - W(x - y) \right) dydz. \end{aligned} \quad (3.2)$$

To prove (3.1), using (3.2) and applying the exponential change of measure we get

$$\begin{aligned} E_x & (e^{-qT}, X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, S_{T-} \leq a, T < \infty) \\ &= \Pi_{\Phi(q)}(-dz - y)e^{-\Phi(q)(-z-x)} \\ &\quad \times \left(\frac{W_{\Phi(q)}(x-b)}{W_{\Phi(q)}(a-b)} W_{\Phi(q)}(a-y) - W_{\Phi(q)}(x-y) \right) dy, \end{aligned}$$

where $W_{\Phi(q)}$ is the 0-scale function of X under $P^{(\Phi(q))}$. Recall that

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x), \quad \Pi_{\Phi(q)}(-dx) = e^{-\Phi(q)x} \Pi(-dx),$$

where $\Pi_{\Phi(q)}$ is the Lévy measure of U under $P^{(\Phi(q))}$, we complete the proof.

Letting $a \rightarrow \infty$ or $b \rightarrow 0$ in (3.1) and using the fact in Zhou [37]

$$\lim_{a \rightarrow \infty} \frac{W^{(q)}(a-y)}{W^{(q)}(a-b)} = e^{-(y-b)\Phi(q)}$$

yield the following result.

Corollary 3.1. *For $q \geq 0$ and for positive numbers x, y, z, a and b such that $b < a \wedge x \wedge y$, $a > x, a > y$,*

$$\begin{aligned} (1). \quad & E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz, I_{T-} > b, T < \infty) \\ &= \pi(-z-y) \left(W^{(q)}(x-b)e^{-(y-b)\Phi(q)} - W^{(q)}(x-y) \right) dydz. \end{aligned} \quad (3.3)$$

$$\begin{aligned} (2). \quad & E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz, S_{T-} \leq a, T < \infty) \\ &= \pi(-z-y) \left(\frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y) \right) dydz. \end{aligned} \quad (3.4)$$

Remark 3.1 Taking derivative with respect to b in (3.3) yields the following result:

$$\begin{aligned} & E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz, I_{T-} \in db, T < \infty) \\ &= \pi(-z-y)e^{-\Phi(q)(y-b)} (W^{(q)'}(x-b) - \Phi(q)W^{(q)}(x-b)) dydzdb. \end{aligned}$$

In the case that X drifts to ∞ , Biffis and Kyprianou [8] also found the result based on a quintuple law in Doney and Kyprianou [13].

Theorem 3.2. *For $q, \beta \geq 0$ and for positive numbers x, y, z, a and b such that $z < b \leq a \wedge x \wedge y$, $a > x, a > y$,*

$$\begin{aligned} & E_x(e^{-qT-\beta(T_0-T)}, X(T-) \in dy, |X(T)| \in dz, I_{T_0} > -b, S_{T_0} \leq a, T_0 < \infty) \\ &= K_x(y, z, a) \frac{W^{(\beta)}(-z+b)}{W^{(\beta)}(b)} dydz, \end{aligned} \quad (3.5)$$

where

$$K_x(y, z, a) = \pi(-z - y) \left(\frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a - y) - W^{(q)}(x - y) \right).$$

Proof Applying the strong Markov property of $\{X(t)\}$ at T we get that

$$\begin{aligned} & E_x(e^{-qT-\beta(T_0-T)}, X(T-) \in dy, |X(T)| \in dz, I_{T_0} > -b, S_{T_0} \leq a, T_0 < \infty) \\ &= E_x(e^{-qT-\beta(T_0-T)}, X(T-) \in dy, |X(T)| \in dz, \\ & \quad \inf_{T \leq t \leq T_0} X(t) > -b, S_{T-} \leq a, T_0 < \infty) \\ &= E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz, S_{T-} \leq a, T < \infty) \\ & \quad \times E_{-z}(e^{-\beta T_0}, I_{T_0} > -b, T_0 < \infty). \end{aligned} \quad (3.6)$$

Applying the exponential change of measure one gets

$$\begin{aligned} E_{-z}(e^{-\beta T_0}, I_{T_0} > -b, T_0 < \infty) &= e^{-\Phi(\beta)z} E_{-z}^{\Phi(\beta)}(I_{T_0} > -b, T_0 < \infty) \\ &= e^{-\Phi(\beta)z} \frac{W_{\Phi(\beta)}(-z + b) - W_{\Phi(\beta)}(-z)}{W_{\Phi(\beta)}(b)} \\ &= \frac{W^{(\beta)}(-z + b)}{W^{(\beta)}(b)}. \end{aligned} \quad (3.7)$$

Now (3.5) follows from (3.4), (3.6) and (3.7). This completes the proof of Theorem 3.2.

Letting $a \rightarrow \infty$ and $b \rightarrow \infty$ in (3.5) yield the following result.

Corollary 3.2. *For $q, \beta \geq 0$ and for positive numbers x, y, z we have*

$$\begin{aligned} & E_x(e^{-qT-\beta(T_0-T)}, X(T-) \in dy, |X(T)| \in dz, T_0 < \infty) \\ &= \pi(-z - y) e^{-\Phi(\beta)z} u^{(q)}(x, y) dy dz. \end{aligned} \quad (3.8)$$

The following result is an extension of Chiu and Yin [11, Theorem 3.2].

Theorem 3.3. *Suppose that the Lévy process X with the Laplace exponent (1.1) drifts to ∞ . Denote by $l = \sup\{t \geq 0 : X(t) < 0\}$ the last passage time below level 0. For $q, \beta \geq 0$ and for positive numbers x, y, z, a and b such that $z < b \leq a \wedge x \wedge y$, $a > x, a > y$, then*

$$\begin{aligned} & E_x(e^{-qT-\beta(l-T)}, X(T-) \in dy, |X(T)| \in dz, I_l > -b, S_l < a, T < \infty) \\ &= K_x(y, z, a) R(z, a, b) dy dz, \end{aligned} \quad (3.9)$$

where

$$K_x(y, z, a) = \pi(-z - y) \left(\frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a - y) - W^{(q)}(x - y) \right),$$

$$\begin{aligned}
R(z, a, b) &= \varphi'(0+)e^{-\Phi(\beta)(b-z)}W^{(\beta)}(b-z) - \varphi'(0+)\Phi'(\beta) \\
&\quad + \varphi'(0+)\frac{W^{(\beta)}(b-z)}{W^{(\beta)}(a+b)}\left(W^{(\beta)}(a) + \Phi'(\beta) - e^{-(a+b)\Phi(\beta)}W^{(\beta)}(a+b)\right).
\end{aligned}$$

Proof The strong Markov property of X yields that the left hand side of (3.9) is equals to

$$\begin{aligned}
&E_x(e^{-qT}, X(T-) \in dy, |X(T)| \in dz, S_T < a, T < \infty) \\
&\times E_{-z}(e^{-\beta l}, I_l > -b, S_l < a, l > 0) := S_x(y, z, a) \times R(z, a, b).
\end{aligned}$$

Letting $\beta \rightarrow 0$ and $b \rightarrow \infty$ in (3.5) yield $S_x(y, z, a) = K_x(y, z, a)$ since $K_x(y, z, \{a\}) = 0$. Applying the strong Markov property of X one finds

$$\begin{aligned}
R(z, a, b) &= E_{-z}(e^{-\beta l}, l < T_a^+, T_{-b}^- = \infty) \\
&= E_{-z}(e^{-\beta l}, T_{-b}^- = \infty) - E_{-z}(e^{-\beta l}, l > T_a^+, T_{-b}^- = \infty) \\
&= E_{-z}(e^{-\beta l}, l > 0) - E_{-z}(e^{-\beta l}, T_{-b}^- < \infty) \\
&\quad - E_{-z}\left(e^{-\beta T_a^+} E_a(e^{-\beta l}, l > 0, T_{-b}^- = \infty), T_a^+ < T_{-b}^-\right) \\
&= E_{-z}(e^{-\beta l}, l > 0) - E_{-z}(e^{-\beta T_{-b}^-} E_{X(T_{-b}^-)}(e^{-\beta l}, l > 0), T_{-b}^- < \infty) \\
&\quad - E_{-z}(e^{-\beta T_a^+}, T_a^+ < T_{-b}^-) E_a(e^{-\beta l}, l > 0, T_{-b}^- = \infty). \tag{3.10}
\end{aligned}$$

Note that

$$\begin{aligned}
&E_a(e^{-\beta l}, l > 0, T_{-b}^- = \infty) = E_a(e^{-\beta l}, l > 0) \\
&\quad - E_a\left(e^{-\beta T_{-b}^-} E_{X(T_{-b}^-)}(e^{-\beta l}, l > 0), T_{-b}^- < \infty\right) \\
&= E_a(e^{-\beta l}, l > 0) - \varphi'(0+)\Phi'(\beta)E_a\left(e^{-\beta T_{-b}^- + \Phi(\beta)X(T_{-b}^-)}, T_{-b}^- < \infty\right) \\
&= E_a(e^{-\beta l}, l > 0) - \varphi'(0+)\Phi'(\beta)P_a^{(\Phi(\beta))}(T_{-b}^- < \infty),
\end{aligned}$$

and

$$E_{-z}(e^{-\beta T_{-b}^-} E_{X(T_{-b}^-)}(e^{-\beta l}, l > 0), T_{-b}^- < \infty) = \varphi'(0+)\Phi'(\beta)P_{-z}^{(\Phi(\beta))}(T_{-b}^- < \infty),$$

where we have used the change of measure argument and the formula

$$E_u(e^{-\beta l}, l > 0) = \varphi'(0+)\Phi'(\beta)e^{\Phi(\beta)u}, u \leq 0.$$

See, Kyprianou [25, Ex. 8.10] or Chiu and Yin [11, (1.5) and Theorem 3.1].

It follows that

$$\begin{aligned}
R(z, a, b) &= E_{-z}(e^{-\beta l}, l > 0) - \varphi'(0+)\Phi'(\beta)P_{-z}^{(\Phi(\beta))}(T_{-b}^- < \infty) \\
&\quad - E_{-z}(e^{-\beta T_a^+}, T_a^+ < T_{-b}^-) \\
&\quad \times \left(E_a(e^{-\beta l}, l > 0) - \varphi'(0+)\Phi'(\beta)P_a^{(\Phi(\beta))}(T_{-b}^- < \infty)\right) \tag{3.11}
\end{aligned}$$

Note that (cf. Kyprianou [25, Ex. 8.10]),

$$E_{-z}(e^{-\beta l}, l > 0) = \varphi'(0+) \Phi'(\beta) e^{-\Phi(\beta)z}, \quad (3.12)$$

$$E_a(e^{-\beta l}, l > 0) = \varphi'(0+) \Phi'(\beta) e^{a\Phi(\beta)} - \varphi'(0+) W^{(\beta)}(a). \quad (3.13)$$

and

$$E_{-z}(e^{-\beta T_a^+}, T_a^+ < T_{-b}^-) = \frac{W^{(\beta)}(b-z)}{W^{(\beta)}(a+b)}. \quad (3.14)$$

Moreover, using a fact in Kyprianou and Palmowski [26] we get

$$P_{-z}^{(\Phi(\beta))}(T_{-b} < \infty) = 1 - \varphi'_{\Phi(\beta)}(0+) W_{\Phi(\beta)}(b-z). \quad (3.15)$$

Since (cf. Pistorius [32])

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x), \quad \varphi_{\Phi(q)}(\lambda) = \varphi(\Phi(q) + \lambda) - q,$$

we can rewrite (3.15) as

$$P_{-z}^{(\Phi(\beta))}(T_{-b} < \infty) = 1 - \varphi'(\Phi(\beta)) e^{-\Phi(\beta)(b-z)} W^{(\beta)}(b-z). \quad (3.16)$$

Inserting (3.12)-(3.16) in (3.11) completes the proof.

The following result generalized the corresponding result in Dos Reis [14] and Zhang and Wu [38] in which the classical compound Poisson risk model and the classical compound Poisson risk model perturbed by Brownian motion were considered, respectively. A different approach can be found in Landriault, Renaud and Zhou [21].

Theorem 3.4. *Suppose that the Lévy process X , with the Laplace exponent (1.1), drifts to ∞ . Let $D = \int_0^\infty 1(X(t) < 0)dt$ denote the total duration for X to stay below 0. Then for $x > 0$, $\beta > 0$,*

$$E_x e^{-\beta D} = \varphi'(0+) \Phi(\beta) e^{\Phi(\beta)x} \int_x^\infty e^{-\Phi(\beta)y} W(y) dy. \quad (3.17)$$

In particular,

$$E_0 e^{-\beta D} = \varphi'(0+) \frac{\Phi(\beta)}{\beta}.$$

Proof The ideas of this proof were partly motivated by Dos Reis [14] and Zhang and Wu [36]. For $\varepsilon \geq 0$, define, with the convention that $\inf \emptyset = \infty$,

$$L_1(\varepsilon) = \inf\{t \geq 0 : X(t) < -\varepsilon\}, \quad R_1(\varepsilon) = \inf\{t \geq L_1(\varepsilon) : X(t) = 0\}.$$

In general, for $k \geq 2$ recursively define

$$L_k(\varepsilon) = \inf\{t \geq R_{k-1}(\varepsilon) : X(t) < -\varepsilon\}, \quad R_k(\varepsilon) = \inf\{t \geq L_k(\varepsilon) : X(t) = 0\}.$$

If there exists some k such that $\{t \geq R_{k-1}(\varepsilon) : X(t) < -\varepsilon\} = \emptyset$, then we define $L_k(\varepsilon) = \infty$ (and consequently $R_k(\varepsilon) = \infty$) and $R_k - L_k = 0$.

We first consider the case where the paths of X are of bounded variation. For convenience we shall write L_k in place of $L_k(0)$ and R_k in place of $R_k(0)$. As the paths of X are of bounded variation, then $0 \leq T = L_1 < R_1 < L_2 < R_2 < \dots$, and $R_k - L_k$ represents the duration of the period of the surplus from k -th below the level 0 to the time that $X(t)$ first visits at 0 after L_k . Thus the random variable D can be decomposed as follows:

$$D = \sum_{k=1}^N (R_k - L_k),$$

where $N = \sup\{k : L_k < \infty\}$ ($N = 0$ if the set is empty). Note that N has a geometric distribution,

$$P_x(N = n) = \begin{cases} R(x), & n = 0, \\ \psi(x)R(0)(\psi(0))^{n-1}, & n = 1, 2, \dots, \end{cases}$$

where $R(x) = 1 - \psi(x)$ and $\psi(x) = P(I_\infty < 0 | X(0) = x)$. The stationarity and independence of increments of X imply that given $N = n$, $\{R_k - L_k, k = 1, \dots, n\}$ are mutually independent and $\{R_k - L_k, k = 2, \dots, n\}$ are identically distributed. A simple argument by using the law of double expectation and the strong Markov property yields

$$\begin{aligned} E_x(e^{-\beta D}) &= P_x(N = 0)E_x(e^{-\beta D} | N = 0) \\ &\quad + \sum_{n=1}^{\infty} P_x(N = n)E_x(e^{-\beta(R_1 - L_1)} | N = n) \\ &\quad \times E_x\left(e^{-\beta \sum_{k=2}^n (R_k - L_k)} | N = n\right) \\ &= P_x(N = 0) + \sum_{n=1}^{\infty} P_x(N = n)E_x(e^{-\beta(R_1 - L_1)} | L_n < \infty, L_{n+1} = \infty) \\ &\quad \times \left\{E_x\left(e^{-\beta(R_2 - L_2)} | L_n < \infty, L_{n+1} = \infty\right)\right\}^{n-1} \\ &= R(x) + R(0)E_x\left(e^{X(T)\Phi(\beta)}, T < \infty\right) \sum_{n=1}^{\infty} \left\{E_0\left(e^{X(T)\Phi(\beta)}, T < \infty\right)\right\}^{n-1} \\ &= R(x) + E_x\left(e^{X(T)\Phi(\beta)}, T < \infty\right) \frac{R(0)}{1 - E_0\left(e^{X(T)\Phi(\beta)}, T < \infty\right)}, \end{aligned} \quad (3.18)$$

where, in the third step, we have used

$$\begin{aligned} E_x(e^{-\beta(R_1 - L_1)} | L_n < \infty, L_{n+1} = \infty) &= E_x(E_{X(T)}e^{-\beta R_1}, T < \infty) / P_x(T < \infty) \\ &= E_x\left(e^{X(T)\Phi(\beta)}, T < \infty\right) / P_x(T < \infty), \end{aligned}$$

and

$$\begin{aligned} E_x(e^{-\beta(R_2-L_2)} | L_n < \infty, L_{n+1} = \infty) &= E_0(E_{X(T)} e^{-\beta R_1}, T < \infty) / P_0(T < \infty) \\ &= E_0(e^{X(T)\Phi(\beta)}, T < \infty) / P_0(T < \infty). \end{aligned}$$

By using (2.9) one finds that

$$E_x(e^{X(T)\Phi(\beta)}, T < \infty) = \beta e^{\Phi(\beta)x} \int_x^\infty e^{-\Phi(\beta)y} W(y) dy - \frac{\beta}{\Phi(\beta)} W(x).$$

The result (3.17) follows, since $R(x) = \varphi'(0+)W(x)$.

Next we consider the case where the paths of X are of unbounded variation. Note that, for $\varepsilon > 0$, $0 \leq L_1(\varepsilon) \leq R_1(\varepsilon) \leq L_2(\varepsilon) \leq R_2(\varepsilon) \leq \dots$, and $R_k(\varepsilon) - L_k(\varepsilon)$ represents the duration of the period of the surplus from k -th below the level $-\varepsilon$ to the time that $X(t)$ first visits at 0 after $L_k(\varepsilon)$. Let

$$D(\varepsilon) = \sum_{k=1}^{N(\varepsilon)} (R_k(\varepsilon) - L_k(\varepsilon)),$$

where $N(\varepsilon) = \sup\{k : L_k(\varepsilon) < \infty\}$ ($N(\varepsilon) = 0$ if the set is empty), which has a geometric distribution,

$$P_x(N(\varepsilon) = n) = \begin{cases} R(x + \varepsilon), & n = 0, \\ \psi(x + \varepsilon)R(\varepsilon)(\psi(\varepsilon))^{n-1}, & n = 1, 2, \dots, \end{cases}$$

where, as before, $R(x) = 1 - \psi(x)$ and $\psi(x) = P(I_\infty < 0 | X(0) = x)$. As above, given $N(\varepsilon) = n$, $\{R_k(\varepsilon) - L_k(\varepsilon), k = 1, \dots, n\}$ are mutually independent and $\{R_k(\varepsilon) - L_k(\varepsilon), k = 2, \dots, n\}$ are identically distributed.

Using the same argument as above we have

$$E_x(e^{-\beta D(\varepsilon)}) = R(x + \varepsilon) + \frac{R(\varepsilon)E_x(e^{X(T_{-\varepsilon}^-)\Phi(\beta)}, T_{-\varepsilon}^- < \infty)}{1 - E_0(e^{X(T_{-\varepsilon}^-)\Phi(\beta)}, T_{-\varepsilon}^- < \infty)}. \quad (3.19)$$

It follows from (2.9) that

$$\lim_{\varepsilon \rightarrow 0} \frac{R(\varepsilon)}{1 - E_0(e^{X(T_{-\varepsilon}^-)\Phi(\beta)}, T_{-\varepsilon}^- < \infty)} = \varphi'(0+) \frac{\Phi(\beta)}{\beta}.$$

From the right continuity of the sample paths of $X(t)$, we have $\lim_{\varepsilon \rightarrow 0} D(\varepsilon) = D$. Thus the result (3.17) follows by letting $\varepsilon \rightarrow 0$ in (3.19) and using (2.9) and $R(x) = \varphi'(0+)W(x)$. This ends the proof.

Remark 3.2 For a spectrally one-sided Lévy process, the double-integral transforms of the duration of stay inside/outside the interval $(0, B)$ ($B > 0$ is a constant) before a fixed time have been obtained by Kadankov and Kadankova [22]. However, our result can not deduced by the known result above.

4 Applications to insurance risk theory

Spectrally negative Lévy processes have been considered recently in Refs. [4, 8, 16, 28, 30, 33], among others, in the context of insurance risk models. Motivated by applications in option pricing and risk management, and inspired recent developments in fluctuation theory for Lévy processes, Biffis and Kyprianou [8] and Biffis and Morales [9] defined an extended version of the Gerber and Shiu expected discounted penalty function introduced by Gerber and Shiu [18]. In addition to the surplus before ruin and the deficit at ruin, it includes the information on the last minimum of the surplus before ruin I_T , where $T = \inf\{t > 0 : X(t) < 0\}$ denoting the ruin time of X . The analysis of the result is mainly based on the quintuple law in Doney and Kyprianou [13].

Motivated by them, we now consider the other generalized version of the Gerber-Shiu expected discounted penalty function:

$$\phi(x; q, w) = E_x(e^{-qT} w(X(T-), |X(T)|, S_{T-}, I_{T-}) 1(T < \infty)), \quad (4.1)$$

where $x \geq 0$ is the initial surplus, $q \geq 0$ can be interpreted as a force of interest, $w : \mathbb{R}^4 \rightarrow [0, \infty)$ is bounded measurable function. Using Theorem 3.1 we get the following corollary:

Corollary 4.1. *Suppose that X drifts to ∞ , $W^{(q)}|_{(0, \infty)} \in C^2(0, \infty)$. Then the function defined in (4.1) can be written as*

$$\phi(x; q, w) = \int_{[0, \infty)^4} w(y, z, a, b) \left\{ K_x^{(q)}(y, z, a, b) + 1_{(\sigma \neq 0)} \delta(y, z, a - x, b) K_6 \right\} dy dz da db,$$

where δ is the multidimensional Dirac Delta function, and

$$K_x^{(q)}(y, z, a, b) = 1(y \geq b, a \geq x, z > 0, b > 0) \pi(-z - y) \sum_{i=1}^5 K_i,$$

$$K_1 = \frac{W^{(q)'}(x - b) W^{(q)'}(a - y)}{W^{(q)}(a - b)},$$

$$K_2 = - \frac{W^{(q)'}(x - b) W^{(q)}(a - y) W^{(q)'}(a - b)}{W^{(q)2}(a - b)},$$

$$K_3 = - \frac{W^{(q)}(x - b) W^{(q)''}(a - b) W^{(q)}(a - y)}{W^{(q)2}(a - b)},$$

$$K_4 = - \frac{W^{(q)}(x - b) W^{(q)'}(a - b) W^{(q)'}(a - y)}{W^{(q)2}(a - b)},$$

$$K_5 = \frac{2W^{(q)'}(a - b) W^{(q)}(a - y) W^{(q)'}(a - b)}{W^{(q)3}(a - b)},$$

$$K_6 = Z^{(q)}(x) - \frac{qW^{(q)}(x)}{\Phi(q)} - \int_0^\infty u^{(q)}(x, y) (\Pi(-y) - \Pi(-\infty)) dy.$$

To end this section we rewrite the generalized Dickson's formula for the Cramér-Lundberg risk process (see Gerber and Shiu [18]) and for jump-diffusion process (see Zhang and Wang [38]) in a more appealing form in terms of the probabilities of ruin or the scale functions.

Consider the jump-diffusion risk process:

$$X(t) = x + ct - \sum_{j=1}^{N(t)} X_j + \sigma B(t), t \geq 0, \quad (4.2)$$

where x is the insurer's initial capital, c is the premium rate, $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ and $\{X_k\}_{k \geq 1}$ are independent random variables with common distribution $P = 1 - \bar{P}$, which has density p , mean μ and $P(0) = 0$, $\{B(t), t \geq 0\}$ is a Brownian motion. Moreover, we assume that $c > \lambda\mu$ and, $\{N(t), t \geq 0\}$, $\{X_k\}_{k \geq 1}$ and $\{B(t), t \geq 0\}$ are assumed to be independent. When $\sigma = 0$, (4.2) is called the Cramér-Lundberg risk process. Those two processes correspond to the cases of spectrally negative Lévy processes with $\Pi\{(-\infty, 0)\} < \infty$ and, with or without Gaussian component. For details of risk theory, we refer the readers to Asmussen [2] and Rolski et al. [34].

Obviously, X is a spectrally negative Lévy process with $Ee^{\alpha(X(t)-x)} = e^{t\varphi(\alpha)}$, where φ is defined as

$$\varphi(\alpha) = c\alpha + \frac{1}{2}\sigma^2\alpha^2 + \lambda\{\hat{p}(\alpha) - 1\}.$$

The following generalized Dickson's formula for the Cramér-Lundberg risk process is due to Gerber and Shiu [18, (6.5) and (6.6)]:

$$f_q(y|x) = \begin{cases} f_q(x|0) \frac{e^{\Phi(q)x - \Psi(x)}}{1 - \Psi(0)}, & y > x \geq 0, \\ f_q(y|0) \frac{e^{\Phi(q)x - \Psi(x-y)}}{1 - \Psi(0)}, & 0 < y \leq x, \end{cases} \quad (4.3)$$

where $\Psi(x) = E_x(e^{-qT + \Phi(q)X(T)} 1(T < \infty))$, and

$$f_q(y|0) = \lambda c^{-1} e^{-\Phi(q)y} (1 - P(y)), \quad \Psi(0) = \lambda c^{-1} \int_0^\infty x e^{-\Phi(q)x} p(x) dx.$$

Using (1.2), we can write $\Psi(x)$ as $\Psi(x) = e^{\Phi(q)x} P_x^{(\Phi(q))}(T < \infty)$. Furthermore, $c(1 - \Psi(0)) = E^{(\Phi(q))} X(1) = \varphi'(\Phi(q))$. As a result, we can rewrite the generalized Dickson's formula (4.3) as

$$f_q(y|x) = \begin{cases} \lambda \bar{P}(y) \frac{e^{\Phi(q)(x-y)}}{\varphi'(\Phi(q))} P_x^{(\Phi(q))}(T = \infty), & y > x \geq 0, \\ \lambda \bar{P}(y) \frac{e^{\Phi(q)(x-y)}}{\varphi'(\Phi(q))} \left(P_x^{(\Phi(q))}(T = \infty) - P_{x-y}^{(\Phi(q))}(T = \infty) \right), & 0 < y \leq x. \end{cases} \quad (4.4)$$

The generalized Dickson's formula for jump-diffusion is due to Zhang and Wang [39]:

$$f_q(y|x) = \begin{cases} \lambda \bar{P}(y) \Phi'(q) e^{-\Phi(q)y} (e^{\Phi(q)x} - M(x)), & y > x \geq 0, \\ \lambda \bar{P}(y) \Phi'(q) e^{-\Phi(q)y} (e^{\Phi(q)x} M(x-y) - M(x)), & 0 < y \leq x, \end{cases} \quad (4.5)$$

where $M(x) = E_x e^{-qT_0}$. From Chiu and Yin [11, Theorem 2.3] we have

$$M(x) = E_x \left(e^{-qT + \Phi(q)X(T)} 1(T < \infty) \right).$$

By (1.2), $M(x) = e^{\Phi(q)x} P_x^{(\Phi(q))}(T < \infty)$. Therefore, (4.5) can be rewritten as

$$f_q(y|x) = \begin{cases} \lambda \bar{P}(y) \frac{e^{\Phi(q)(x-y)}}{\varphi'(\Phi(q))} P_x^{(\Phi(q))}(T = \infty), & y > x \geq 0, \\ \lambda \bar{P}(y) \frac{e^{\Phi(q)(x-y)}}{\varphi'(\Phi(q))} \left(P_x^{(\Phi(q))}(T = \infty) - P_{x-y}^{(\Phi(q))}(T = \infty) \right), & 0 < y \leq x, \end{cases} \quad (4.6)$$

where we have used $\Phi'(q) \cdot \varphi'(\Phi(q)) = 1$.

Remark 4.1 The results (4.4) and (4.6) can also be expressed in terms of the scale functions by using (1.2). We remark that the positive safety loading condition is not required in the case $q > 0$. Thus the corresponding conditions can be removed here.

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